Adjoint of Unbounded Operators

P. Sam Johnson

Outline of the talk

In the lecture, we define adjoint of unbounded linear operators on Hilbert spaces and discuss some results on adjoints.

Notations

- H, an infinite-dimensional Hilbert space (not necessarily separable) over the field $\mathbb{K}$ of real or complex scalars.
- $D(T)$, the domain of an operator T
- $R(T)$, the range of T
- $N(T)$, the null space of T
- $L_2[0,1]$, the space of all square-integrable functions on $[0,1]$
- $AC[0,1]$, the space of all absolutely continuous functions on $[0,1]$
- ℓ_2, the space of all square-summable sequences

Adjoint of an operator with dense domain

Let $T : D(T) \to H$ be a linear operator. If S is a linear operator such that for all $x \in D(T)$ and $y \in D(S)$

$$\langle Tx, y\rangle = \langle x, Sy\rangle, \tag{1}$$

then S is called a **formal adjoint** of T.

The operator S_0 such that $D(S_0) = \{0\}$ is a formal adjoint of every operator. So we look for an operator satisfying (1) with a maximal domain, and such operator should be uniquely defined.

LA-2(P-85)N-1

Recall : Bounded case

Theorem 1.

Let H and K be Hilbert spaces and $T \in B(H, K)$. Then there is a unique $T^* \in B(K, H)$ such that

$$\langle Tx, y\rangle_K = \langle x, T^*y\rangle_H \quad \text{for all } x \in H, y \in K.$$

The operator T^* is called the **adjoint** *of T.* *LA-2(P-15)T-14*

In general, a bounded linear operator on an inner product space need not have an adjoint. The fact that the completeness is essential in Theorem 1.

Recall : Bounded case

Exercise 2.

Let H be a Hilbert space and $T : H \to H$ be a linear map. Suppose that there is a linear map $S : H \to H$ such that

$$\langle Tx, y \rangle = \langle x, Sy \rangle \quad \text{for all } x, y \in H.$$

Then both operators T and S are bounded and $S = T^*$. LA-2(P-27)E-21

Adjoint of an operator with dense domain

Exercise 3.

If T is bounded and if the relation (1) holds for all x,y in H, then S would be the uniquely defined bounded operator, called the "adjoint of T."

However, in the unbounded case (1) by itself, does not define S uniquely.

It is possible although not obvious that of all the operators satisfying (1) there will be one with a domain **which is maximal** (in the sense of set inclusion). It is this operator, T^* say, which provides the required generalization of the adjoint provided $D(T)$ is dense in H.

Adjoint of an operator with dense domain

Let T be a densely defined operator on H. The choice of $D(T^*)$ is clarified as follows:

Let $D(T^*)$ be the set of $y \in H$ such that there exists an z in H with

$$\langle Tx, y\rangle = \langle x, z\rangle \quad \text{for all} \quad x \in D(T).$$

Given y, the element z is uniquely determined, as $D(T)$ is dense, for if there is a $\tilde{z}$ such that

$$\langle Tx, y\rangle = \langle x, \tilde{z}\rangle, \quad \text{then} \quad \langle x, z - \tilde{z}\rangle = 0,$$

we get $z = \tilde{z}$. (Note that unless $D(T)$ is dense, this definition does not make sense).

Now set $z = T^*y$. It is easy to check that T^* is linear, and clearly

$$\langle Tx, y\rangle = \langle x, T^*y\rangle, \quad \text{for all} \quad x \in D(T) \quad \text{and} \quad y \in D(T^*).$$

Adjoint of an operator with dense domain

Thus (1) is satisfied with $S = T^*$, and further every S satisfying this equation is a restriction of T^*. Therefore, as asserted above, $D(T^*)$ is maximal.

In most cases of interest, $D(T^*)$ itself is dense in H.

Adjoint of an operator with dense domain

Definition 4.

Suppose T is a linear operator from H into H with dense domain. Let $D(T^*)$ be the set of all elements y such that there is an z with

$$\langle Tx, y\rangle = \langle x, z\rangle \quad \text{for all } x \in D(T).$$

Let T^* be the operator with domain $D(T^*)$ and with $T^*y = z$ on $D(T^*)$ or equivalently assume that

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \quad \text{for all } x \in D(T), y \in D(T^*).$$

T^* is called the **adjoint** *of T.*

Adjoint of an operator with dense domain

Exercise 5.

Show that $D(T^*)$ is the set of all elements $y \in H$ such that the linear functional $x \mapsto \langle Tx, y\rangle$ is continuous (bounded) on $D(T)$. That is, $D(T^*) = \{y \in H : \textit{the functional } x \mapsto \langle Tx, y\rangle \textit{ is continuous on } D(T)\}$.

The denseness of domain is necessary and sufficient for existence of the adjoint. That is, T^* **exists iff** $D(T)$ **is dense in** H**.**

Example 6.

Let $H = L_2[0,1]$. *Define* $T : H \to H$ *by* $Tf = f'$ *with*

$$D(T) = \left\{ f \in H : f \in AC[0,1], f' \in H, f(0) = f(1) = 0 \right\}.$$

Show that $D(T)$ *is dense and find* T^* *by giving its domain and action.*

LA-2(P-88)E-2

Properties of adjoints

Exercise 7.

Let S and T be two densely defined operators on H. Show the following.

1. $(\alpha T)^* = \overline{\alpha} T^* \quad \forall\, \alpha \in \mathbb{C}$.
2. *If* $T \subset S$, *then* $S^* \subset T^*$.
3. *If* $D(S+T)$ *is dense in* H, *then* $S^* + T^* \subset (S+T)^*$.
4. *If* $D(ST)$ *is dense, then* $T^* S^* \subset (ST)^*$.
5. *If* S *is an everywhere defined bounded operator, then* $(S+T)^* = S^* + T^*$ *and* $(ST)^* = T^* S^*$.
6. $N(T^*) = R(T)^{\perp}$.
7. $R(T^*) \subseteq N(T)^{\perp}$.

LA-2(P-91)T-3

Adjoint of an operator with dense domain

Note that $H \times H$ is naturally equipped with the inner product

$$\Big\langle (x,y),(x',y') \Big\rangle_{H\times H} = \langle x, x' \rangle_H + \langle y, y' \rangle_H$$

which makes it a Hilbert space. Define

$$U(x,y) = (y,-x) \quad \text{and} \quad V(x,y) = (y,x) \quad (x \in H, y \in H).$$

1. U and V are isomorphisms from $H \times H$ onto $H \times H$ with $U^2 = -\mathbb{I}$ and $V^2 = \mathbb{I}$.
2. U^{-1} and V^{-1} are defined by

$$U^{-1}(x,y) = (-y,x) \quad \text{and} \quad V^{-1}(x,y) = (y,x) \quad (x \in H, y \in H).$$

LA-2(P-95)R-5

Proposition 8.

If T is closed and injective, then T^{-1} is closed. *LA-2(P-95)P-6*

A remarkable description of T^* in terms of T

The following result tells that once $G(T^*)$ is known, so are $D(T^*)$ and T^*.

Theorem 9.

If T is a densely defined operator on H, then

1. $G(T^*) = U[G(T)^\perp] = [UG(T)]^\perp$, *(the orthogonal complement of $UG(T)$ in $H \times H$.)* LA-2(P-96)T-7
2. $G(T)^\perp = U^{-1}[G(T^*)]$. LA-2(P-97)T-7A

- In general, $G(T^*)^\perp \neq U[G(T)]$. But if T is closed, then $G(T^*)^\perp = U[G(T)]$.
- Moreover, $H \times H = \overline{G(T)} \oplus UG(T^*)$.

T^* is more stable than T

Theorem 10.

Let T be a densely defined operator (may not be even closable). Then the operator T^* is closed. LA-2(P-97)T-8

One may say that the construction of the adjoint operator produces an operator which is **more stable** than T because T^* is always closed (irrespectively of whether T is or not).

Corollary 11.

Let T be a densely defined operator on H. If T is closable, then

1. $(\overline{T})^* = T^*$ *(T and $\overline{T}$ have the same adjoint).* LA-2(P-98)T-9
2. $N(\overline{T}) = R(T^*)^\perp$. LA-2(P-94)P-4

Theorem 12.

If T is a densely defined closed operator on H, then

$$H \times H = UG(T) \oplus G(T^*),$$

a direct sum of two orthogonal subspaces. LA-2(P-98)T-10

We proved that if T is densely defined, then $G(T^*) = [UG(T)]^\perp$. But in general, $G(T^*)^\perp \neq UG(T)$. When T is closed, $G(T^*)^\perp = UG(T)$.

Relation between adjoints and inverses

Exercise 13.

Let T be a densely defined operator on H. Show the following.

1. *If T is injective and closed, then T^* is injective and* $(T^{-1})^* = (T^*)^{-1}$. *LA-2(P-98)T-11*
2. *If T is injective and invertible (T^{-1} is bounded), then T^* is injective and* $(T^{-1})^* = (T^*)^{-1}$. *LA-2(P-99)T-12*
3. *If T is injective and $R(T)$ is dense in H, then T^* is injective and* $(T^{-1})^* = (T^*)^{-1}$. *LA-2(P-100)T-13*

Results on adjoints

If $D(T^*)$ happens to be dense in H, then the operator T must be closable, as the result follows. Also, T^{**} is a natural (minimal) closed extension of T, i.e., $\overline{T} = T^{**}$.

Theorem 14.

Let $T : D(T) \to H$ be a densely defined operator on H. Then T is closable if and only if T^* is densely defined, in which case $\overline{T} = (T^*)^*$. LA-2(P-101)T-14

That is, denseness of domain of T^* is a necessary and sufficient condition for an operator T to be closable.

Results on adjoints

Theorem 15.

Let T be an everywhere defined operator on H such that $D(T^*)$ is dense in H. Then T is bounded. LA-2(P-102)E-15

Theorem 16.

Let T be a densely defined closed operator in H. Then $D(T^*)$ is dense and $T^{**} = T$. LA-2(P-102)E-15

For closable operators, we proved that $(\overline{T})^* = T^*$ and $\overline{T} = (T^*)^*$, hence the operation $*$ behaves like $\perp$ in inner product space.

The domain of the adjoint can be quite small.

Exercise 17.

For every $k \in \mathbb{N}$, let the sequence $\{n_{k,\ell}\}_{\ell=1}^{\infty}$ of $\mathbb{N}$ be chosen in such a way that

$$\{n_{k,\ell} : \ell \in \mathbb{N}\} \cap \{n_{j,\ell} : \ell \in \mathbb{N}\} = \emptyset \quad \text{for } j \neq k$$

and

$$\bigcup_{k \in \mathbb{N}} \{n_{k,\ell} : \ell \in \mathbb{N}\} = \mathbb{N}.$$

With these sequences, let us define the operator T on ℓ_2 with $D(T) = c_{00}$ by

$$T(f) = \left(\sum_{\ell=1}^{\infty} f_{n_{1,\ell}}, \sum_{\ell=1}^{\infty} f_{n_{2,\ell}}, \dots \right).$$

Show that T^* exists but $D(T^*) = \{0\}$.

Ajoint of several operators is one operator.

Example 18.

Consider the operator, for $k \in \mathbb{N}$ $T_k = -i\frac{d}{dt}$ *on the domain* $D(T_k) = \{f \in C^k[0,1] : f(0) = f(1) = 0\}$ *with the action* $T_k : C^k[0,1] \to L_2[0,1]$. *Prove the following :*

1. $T_1 \supset T_2 \supset T_3 \supset \cdots$ *(Hint :* $C^1[0,1] \supset C^2[0,1] \supset C^3[0,1] \ldots$*).*
2. *None of them is closed.*
3. *Each* T_k *is closable.*
4. *But closures of all* T_k *are the same;* $\overline{T_1} = \overline{T_2} = \overline{T_3} = \cdots = \overline{T} = -i\frac{d}{dt}$ *on the domain* $D(\overline{T}) = \{f \in AC[0,1] : f(0) = f(1) = 0\}$.
5. $T_1^* = T_2^* = \cdots = T^* = -i\frac{d}{dt}$ *on the domain* $\left\{f \in L^2[0,1] : f, f' \in AC[0,1]\right\}$.

Irrespectively of which T_k, one starts from, it is T^* the **important** operator, which in turn determines the closure $\overline{T}$, via $T^{**} = \overline{T}$.

Results on Closed Operators (analogous to bounded operators)

Theorem 19.

Let $T : D(T) \to H$ be a densely defined closed operator on H. Then $R(T)$ is closed iff $T|_{D(T)\cap N(T)^{\perp}}$ is bounded from below. LA-2(P-106)T-18

Theorem 20.

Let $T : D(T) \to H$ be a densely defined closed operator on H. Then $R(T)$ is closed iff $R(T^*)$ is closed. LA-2(P-107)T-19

Theorem 21.

Let $T : D(T) \to H$ be a densely defined closed operator on H and M be a closed subspace of H containing $R(T)$. Then $T^*|_{D(T^*)\cap M}$ is bounded from below iff $M = R(T)$.

Summary

- T is densely defined iff T^* exists.
- Let T be a **densely defined** (T^* exists). Then the following statements are true:
  1. T^* is closed.
  2. T^* is densely defined iff T is closable, in this case $\overline{T} = T^{**}$.
  3. $N(T^*) = R(T)^\perp$.
  4. $G(T^*) = [UG(T)]^\perp = U[G(T)^\perp]$.
     Applying U^{-1}, we get $G(T)^\perp = U^{-1}G(T^*)$ (since U^{-1} is isometry and preserves orthogonality).
  5. $D(T^*) = \{0\}$ iff $G(T)$ is dense in $H \times H$.
  6. If T is injective and invertible, then $(T^{-1})^* = (T^*)^{-1}$.
  7. $H \times H = \overline{G(T)} \oplus UG(T^*) = U^{-1}\overline{G(T)} \oplus G(T^*)$.
  8. $H \times H = G(T^*) \oplus UG(T^{**}) = UG(T^*) \oplus G(T^{**})$ (since $G(T^*)$ is always closed.)

Summary

- Let T be densely defined **closable** (T^* is densely defined). Then
  1. $(\overline{T})^* = T^*$.
  2. $\overline{T} = T^{**}$.
  3. $N(\overline{T}) = R(T^*)^{\perp}$.

Summary

Let T be densely defined **closed**. Then

1. $H \times H = UG(T) \oplus G(T^*)$. That is, $G(T^*)^\perp = UG(T)$.
2. $D(T^*)$ is dense and $T^{**} = T$.
3. $N(T) = R(T^*)^\perp$.
4. $N(T)$ is a closed subspace of H.
5. $R(T)$ is closed iff $R(T^*)$ is closed.
6. $R(T)$ is closed iff $T|_{D(T)\cap N(T)^\perp}$ is bounded from below.
7. If T is injective, then T^{-1} is closed.
8. If T is injective, then $(T^{-1})^* = (T^*)^{-1}$.

References

- Joachim Weidmann, *Linear Operators in Hilbert Spaces*, Springer, (1980) (pages mainly from 88 to 91).
- Walter Rudin, *Functional Analysis - Second Edition*, McGraw-Hill International Editions, (1991) (pages mainly from 347 to 356).
- Israel Gohberg, Seymour Goldberg and Marinus A. Kaashoek, *Basic Classes of Linear Operators*, Birkhauser Verlag, Berlin, 2000 (pages mainly from 208 to 210).